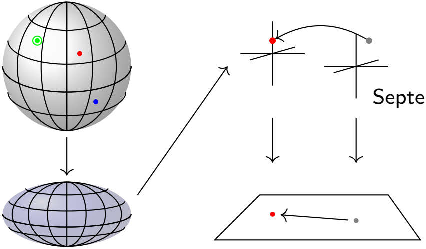
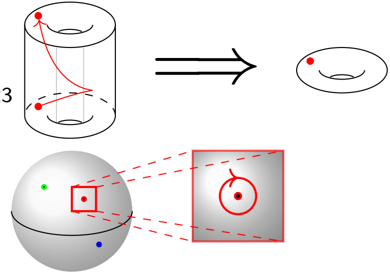
Dynamic CS/WZNW duality: the missing case

The equivalence of the Hitchin and Knizhnik–Zamolodchikov connections

Tim Henke
Joint work with Jørgen Ellegaard Andersen



September 19, 2023



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Chern–Simons Theory: Recap

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3-dimensional gauge theory on M_3
G compact, simply connected gauge group (e.g. $\mathrm{SU}(N)$)
$\mathfrak{g} = \mathrm{Lie}(G)$ and $[\xi_a, \xi_b] = f^c_{ba}\xi_c$

Field content

Fields: connections $A \in \Omega^1(M_3; \mathfrak{g}) + \mathrm{A}_0$
$A = \xi_a A^a_\mu \, \mathrm{d}x^\mu$ Lie algebra-valued 1-form

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Classical solutions

Moduli space of flat connections

Chern–Simons Theory: Reduction to boundary

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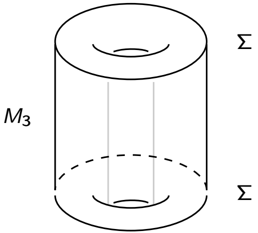

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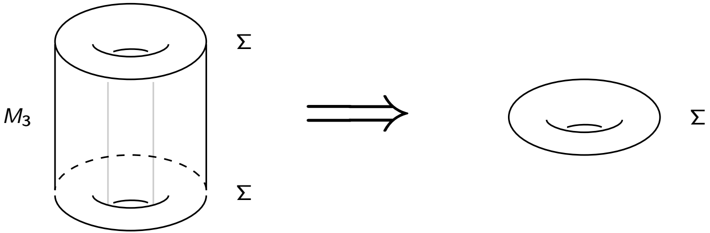
2-dim. gauge theory on Σ_2 closed surface
Moduli space $\mathcal{M}(\Sigma)$ of flat connections on Σ



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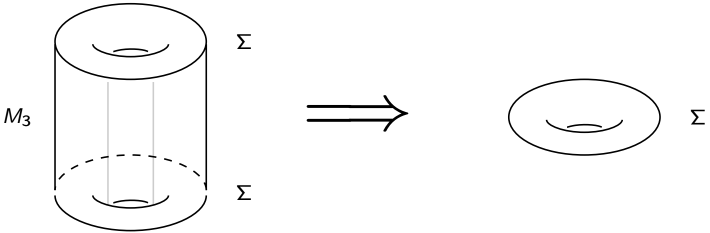

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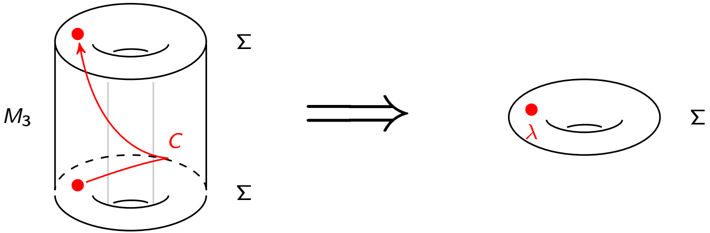

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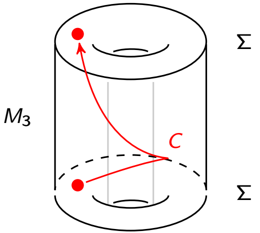
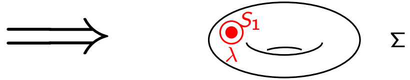
Moduli space

$\mathcal{M}(\Sigma, \vec{p}, \vec{\lambda})$ of flat connections on
$\Sigma^\circ := \Sigma \setminus \{p_1, \ldots, p_n\}$
$\int_{S_i} A \sim \exp(2\pi i \lambda_i / k)$
$\cong$ Moduli space of $\vec{\lambda}$-parabolic G-bundles





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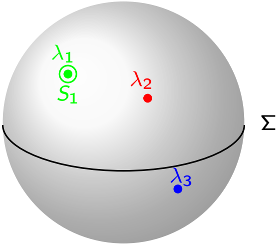

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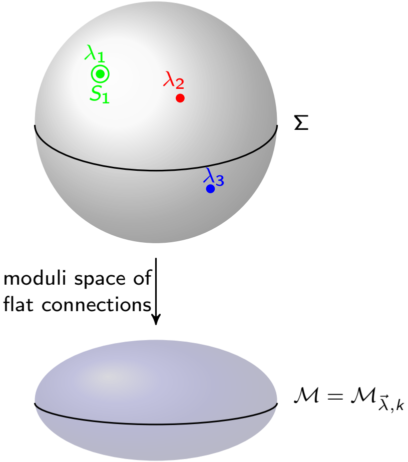

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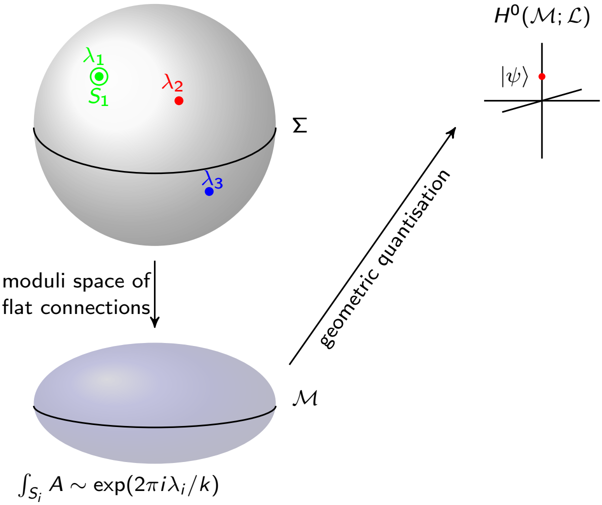

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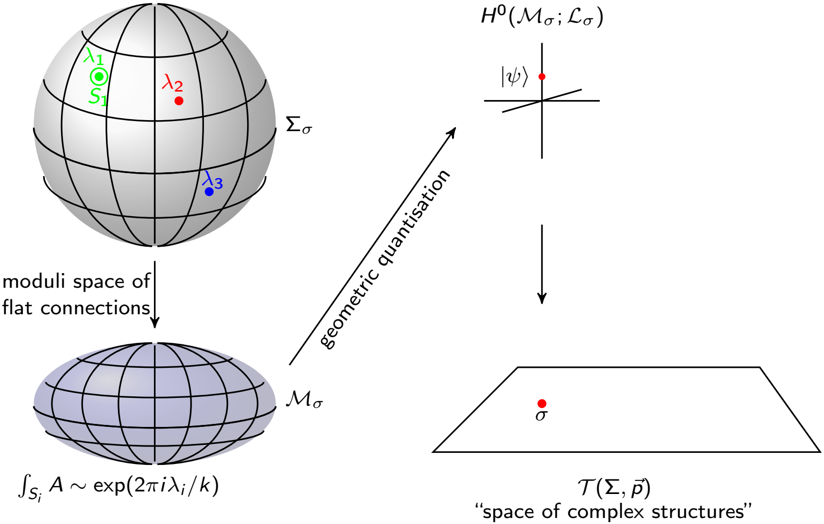

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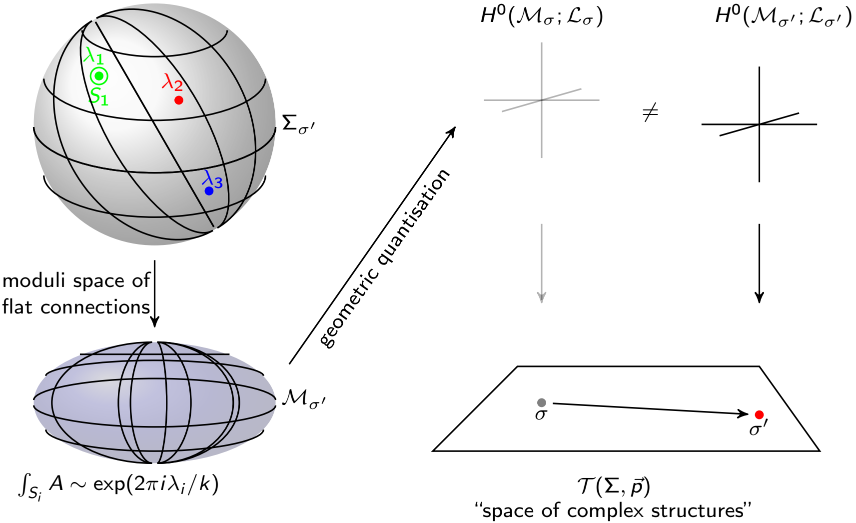

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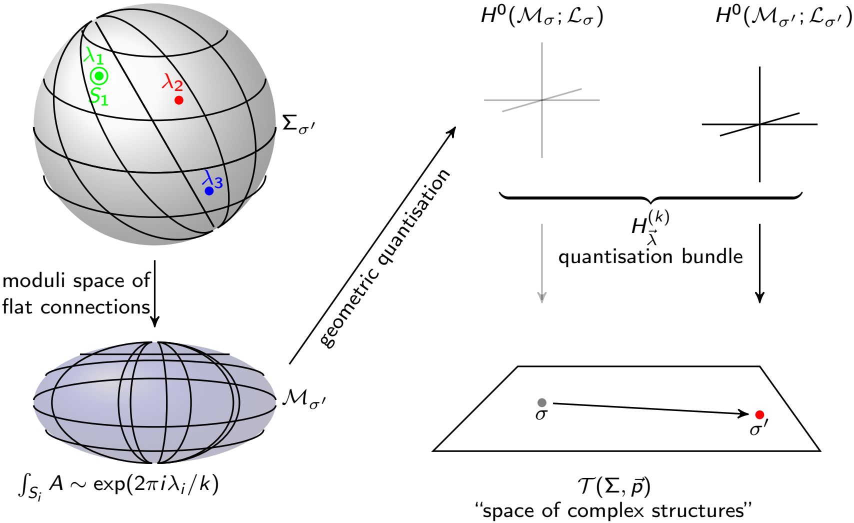

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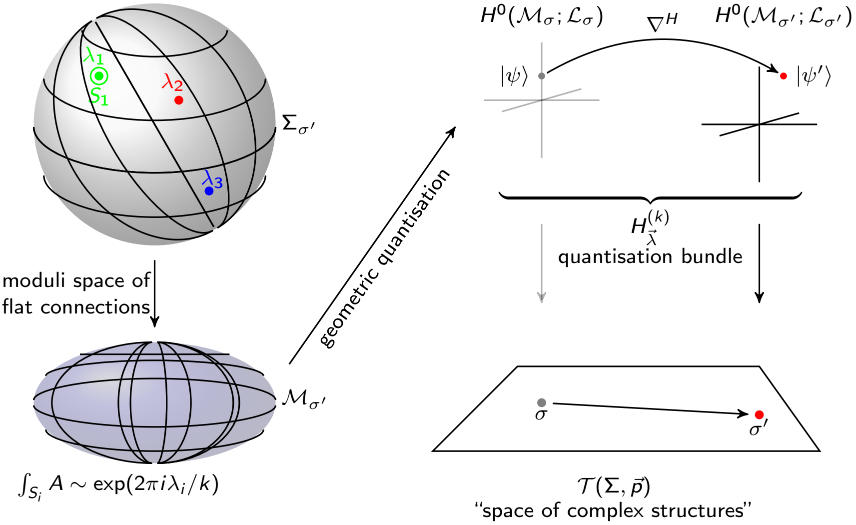

Chern–Simons Theory: The metaplectic-corrected Hitchin Connection

Hitchin connection [Axelrod et al., 1991, Hitchin, 1990]

Quantisation bundle $H^{(k)}_{\vec{\lambda}} \twoheadrightarrow \mathcal{T}(\Sigma, \vec{p})$
space of complex structures
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$H^1(M) = 0$

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Lemma (Andersen and H., to appear)

The metaplectic-corrected Hitchin connection exists on $H^{(k)}_{\vec{\lambda}}$ for regular weights $\vec{\lambda}$.

Lemma (Andersen and H., to appear)

For every tuple $\vec{\lambda}$ the sum $\vec{\lambda} + \vec{\rho}$ is regular and the inclusion $H^{(k)}_{\vec{\lambda}} \subseteq {}^{\delta}H^{(k+h^\vee)}_{\vec{\lambda}+\vec{\rho}}$ is preserved by the metaplectic-corrected Hitchin connection.

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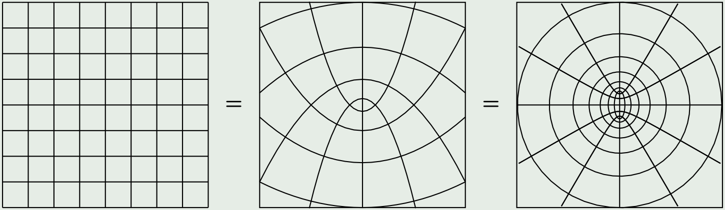

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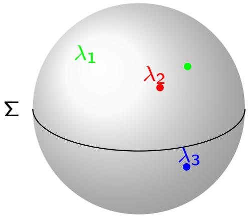

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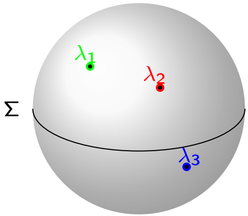

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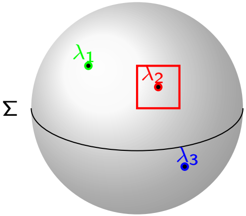

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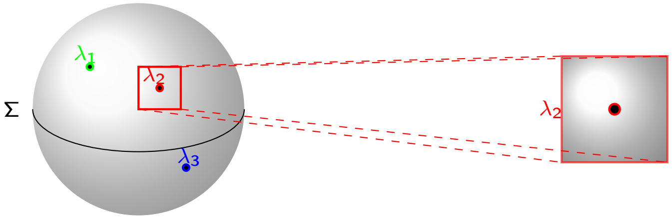

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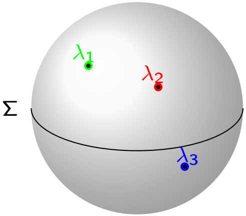

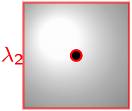

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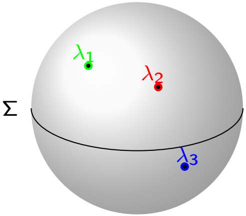

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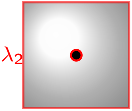

"primary field insertions" $\Phi^{\lambda_i}(p_i)$

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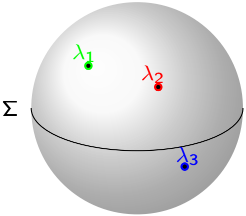

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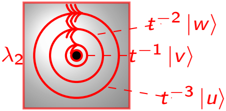

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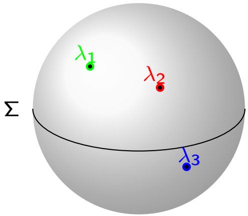

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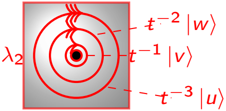

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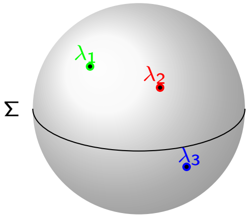

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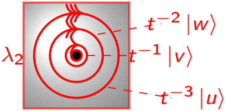

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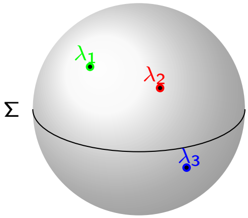

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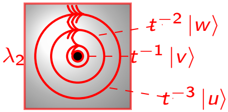
highest weight λ, central charge k.



"primary field insertions" $\Phi^{\lambda_i}(p_i)$
t^{-1} creation, t annihilation
$\mathfrak{g} \circlearrowright V_\lambda$ irr. rep. highest weight λ

The Tsuchiya–Ueno–Yamada/Wess–Zumino–Novikov–Witten model

TUY/WZNW CFT

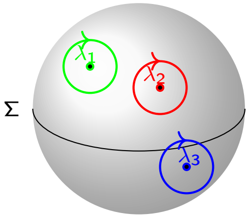
2-dimensional gauge theory on Σ_2
G compact, connected, simply connected gauge group, $\mathfrak{g} = \mathrm{Lie}(G)$
Conformal field theory

Citations

[Wess and Zumino, 1974, Tsuchiya et al., 1989, Ueno, 2008, Kawamoto et al., 1988, Andersen and Ueno, 2007b]

Conformal blocks (conformal covacua)

$\mathcal{V}_{\vec{\lambda},k} := \left[\bigotimes_{i=1}^n \mathscr{H}_k(\lambda_i)\right]_{\mathfrak{g}[\Sigma^\circ]}$ where $\mathfrak{g}[\Sigma^\circ] = \{\Sigma^\circ \to \mathfrak{g} \text{ algebraic}\} \circlearrowright \mathscr{H}_k(\lambda)$ via Laurent exp.



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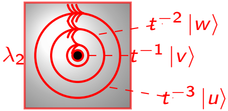

"primary field insertions" $\Phi^{\lambda_i}(p_i)$
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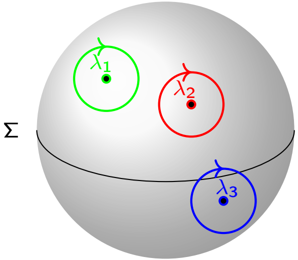
WZNW CFT: Sheaf of Conformal Blocks



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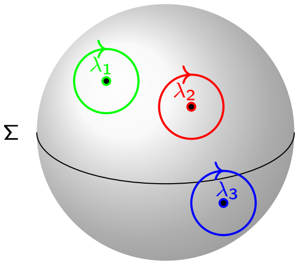

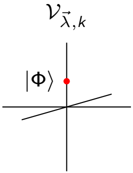

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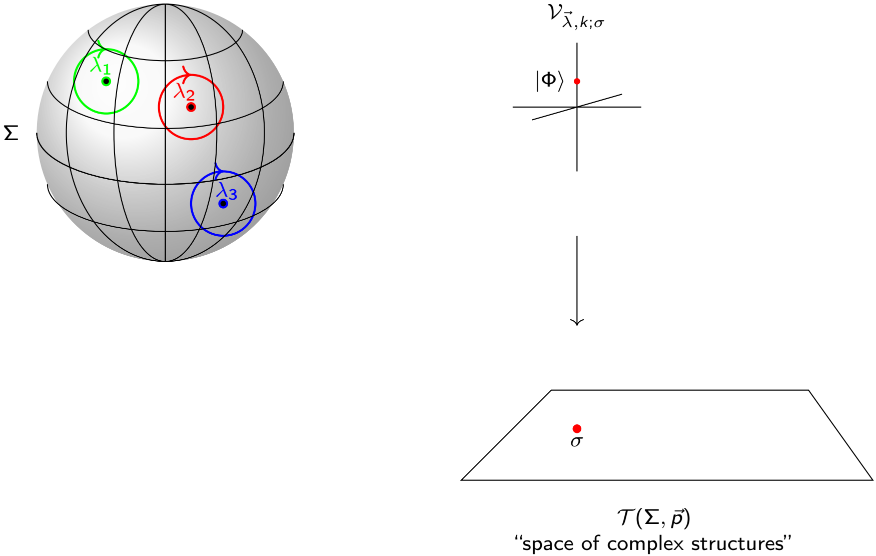
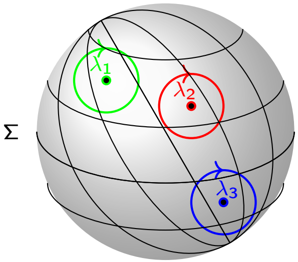
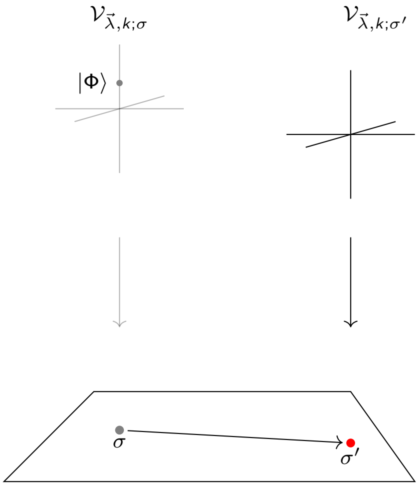


WZNW CFT: Sheaf of Conformal Blocks





$\mathcal{T}(\Sigma, \vec{p})$
"space of complex structures"

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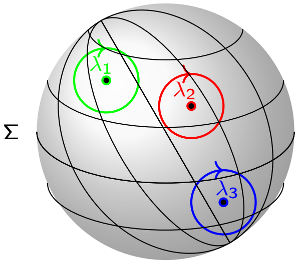

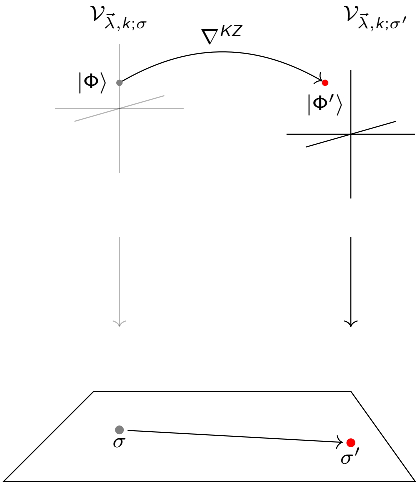

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TUY CFT: Knizhnik–Zamolodchikov Connection

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Knizhnik–Zamolodchikov equations [Knizhnik and Zamolodchikov, 1984]

$$\text{Ward Identities} \implies \left((k + h^\vee) \frac{\partial}{\partial z_i} + \sum_{i \neq j} \sum_a \frac{\xi_i^a \otimes \xi_j^a}{z_i - z_j} \right) \langle \Phi(v_1, z_1) \cdots \Phi(v_N, z_N) \rangle = 0$$

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Knizhnik–Zamolodchikov connection

$$\nabla^{KZ} := \mathrm{d} + \frac{1}{2(k + h^\vee)} \sum_{i \neq j} \frac{\xi_i^a \otimes \xi_j^a}{z_i - z_j} (\mathrm{d}z_i - \mathrm{d}z_j)$$

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Higher genus: Knizhnik–Zamolodchikov–Bernard equations/Tsuchiya–Ueno–Yamada connection [Bernard, 1988, Tsuchiya et al., 1989]

CS = TUY

Data

$p_1, \ldots, p_n \in \Sigma$ smooth surface, $\Sigma^\circ := \Sigma \setminus \{p_1, \ldots, p_n\}$
$G(= \mathrm{SU}(N))$ semi-simple, connected, simply connected Lie group, $\mathfrak{g} = \mathrm{Lie}(G)(= \mathfrak{su}(N))$
$\lambda_1, \ldots, \lambda_n \in \mathfrak{g}^*$ representations: integral, dominant weights/charges
$k > 0$ integer: level/quantisation parameter

	CS	WZNW
Physics	Quantised classical gauge theory	Conformal gauge theory
p_i	Punctures	Poles
Fields	Flat $\mathfrak{g}$-connections on Σ°	$\mathfrak{g}$-modes on Σ°
Charges	Holonomy $p_i \sim \lambda_i^\vee / k$	Highest weight λ_i, central charge k
Dependence	Complex structure	Conformal structure
Connection	Hitchin connection	KZ connection

Theorem (Uniformisation Theorem [Witten, 1989, Laszlo and Sorger, 1997])

$$H^{(k)}_{\vec{\lambda}} \cong \mathcal{V}_{\vec{\lambda},k} \text{ as vector bundles over } \mathcal{T}(\Sigma, \vec{p}).$$

Main Theorem

Theorem (Uniformisation Theorem)

$H^{(k)}_{\vec{\lambda}} \cong \mathcal{V}_{\vec{\lambda},k}$ as vector bundles over $\mathcal{T}(\Sigma, \vec{p})$.

Previous work

[Laszlo, 1998], [Andersen et al., 2017], [Biswas et al., 2021b, Biswas et al., 2021a], [Daskalopoulos and Wentworth, 2011]

Theorem (Andersen and H., to appear)

Let $p_1, \ldots, p_n \in \Sigma = \mathbb{CP}^1$ be a smooth pointed surface and let $k > 0$, $n \geq 3$, and $N \geq 2$ be integers. Let $G = \mathrm{SU}(N)$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be integral, dominant weights for G. If $n + N \geq 7$, then the uniformisation isomorphism

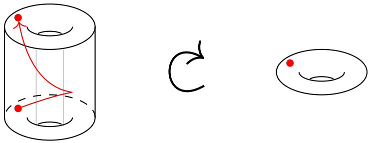
$$H^{(k)}_{\vec{\lambda}}(\Sigma, \vec{p}) \cong \mathcal{V}_{\vec{\lambda},k}(\Sigma, \vec{p})$$

of vector bundles over Teichmüller space $\mathcal{T}(\Sigma)$ is projectively flat with respect to the Hitchin connection ∇^H and the Knizhnik–Zamolodchikov connection ∇^{KZ}.

Application: Topological Quantum Field Theory

To do:

Construct full gauge-theoretic WRT TQFT from this via [Andersen and Petersen, 2016].



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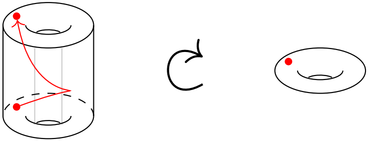

	Gauge theory	Conformal field theory	Braid algebra
Physics	Chern–Simons QFT [Witten, 1989]	WZNW CFT [Witten, 1989]	Algebraic [Witten, 1989]
Maths	?	TUY TQFT [Andersen and Ueno, 2007b]	Modular tensor categories [Reshetikhin and Turaev, 1991]

Theorem ([Andersen and Ueno, 2007a, Andersen and Ueno, 2012, Andersen and Ueno, 2015])

The Tsuchiya–Ueno–Yamada TQFT constructed in [Andersen and Ueno, 2007b] is isomorphic to the Witten–Reshetikhin–Turaev TQFT.

Proof idea

Proof sketch.

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1: geometrise Knizhnik–Zamolodchikov connection

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$\mathcal{V}_{\vec{\lambda},k} \subseteq V_{\vec{\lambda}}^{\mathrm{inv}}$

$V_{\vec{\lambda}} \cong H^0(L_{\vec{\lambda}} \twoheadrightarrow F_{\vec{\lambda}})$ Bott–Borel–Weil Theorem, $F_{\vec{\lambda}} = \prod_i G/P_{\lambda_i}$

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$\mathcal{M}_{\text{flat}} \cong \mathcal{M}_{\text{parabolic}}$

$\mathcal{M}_{\text{parabolic}} \supseteq U \subseteq F_{\vec{\lambda}}/\!\!/G$ open+dense

Proof idea

Proof sketch.

1: geometrise Knizhnik–Zamolodchikov connection

$\mathcal{V}_{\vec{\lambda},k} \subseteq V^{\text{inv}}_{\vec{\lambda}}$

$V^{\text{inv}}_{\vec{\lambda}} \cong H^0((L_{\vec{\lambda}} \twoheadrightarrow F_{\vec{\lambda}}) /\!\!/ G)$ Bott–Borel–Weil Theorem, $F_{\vec{\lambda}} = \prod_i G/P_{\lambda_i}$

$\nabla^{KZ}_V = \mathrm{d} + u(V)$ with $u(V)$ differential operator.

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$\mathcal{M}_{\text{flat}} \cong \mathcal{M}_{\text{parabolic}}$

$\mathcal{M}_{\text{parabolic}} \supseteq U \subseteq F_{\vec{\lambda}} /\!\!/ G$ open+dense

$\mathcal{L}|_U \cong (L_{\vec{\lambda}} /\!\!/ G)|_U$

Proof idea

Proof sketch.

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$H^0(\mathcal{L}\ ) \cong H^0(L_{\vec{\lambda}}\ /\!\!/G) \subseteq H^0(\mathcal{L}^{\ell}|_U)$ preserved by ∇^H and ∇^{KZ}

Proof idea

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ℓ-independence $\implies$ eliminate higher orders $\implies$ $\nabla^H - \nabla^{KZ}$ scalar

□

References I

Andersen, J. E. (2012).
Hitchin's connection, Toeplitz operators, and symmetry invariant deformation quantization.
Quantum topology, 3(3):293–325.

Andersen, J. E., Himpel, B., Jørgensen, S. F., Martens, J., and McLellan, B. (2017).
The Witten–Reshetikhin–Turaev invariant for links in finite order mapping tori I.
Advances in Mathematics, 304:131–178.

Andersen, J. E. and Petersen, W. (2016).
Construction of modular functors from modular tensor categories.
arXiv preprint arXiv:1602.04907.

Andersen, J. E. and Ueno, K. (2007a).
Abelian conformal field theory and determinant bundles.
International Journal of Mathematics, 18(08):919–993.

Andersen, J. E. and Ueno, K. (2007b).
Geometric construction of modular functors from conformal field theory.
Journal of Knot theory and its Ramifications, 16(02):127–202.

References II

Andersen, J. E. and Ueno, K. (2012).
Modular functors are determined by their genus zero data.
Quantum Topology, 3(3):255–291.

Andersen, J. E. and Ueno, K. (2015).
Construction of the witten–reshetikhin–turaev tqft from conformal field theory.
Inventiones mathematicae, 201(2):519–559.

Axelrod, S., Della Pietra, S., and Witten, E. (1991).
Geometric quantization of chern simons gauge theory.
representations, 34:39.

Bernard, D. (1988).
On the wess-zumino-witten models on the torus.
Nuclear Physics B, 303(1):77–93.

Biswas, I., Mukhopadhyay, S., and Wentworth, R. (2021a).
Geometrization of the tuy/wzw/kz connection.
arXiv preprint arXiv:2110.00430.

Biswas, I., Mukhopadhyay, S., and Wentworth, R. (2021b).
Ginzburg algebras and the hitchin connection for parabolic g-bundles.
arXiv preprint arXiv:2103.03792.

References III

- Daskalopoulos, G. D. and Wentworth, R. A. (2011).
  Geometric quantization for the moduli space of vector bundles with parabolic structure.
  In *Geometry, topology and physics*, pages 119–156. De Gruyter.
- Hitchin, N. J. (1990).
  Flat connections and geometric quantization.
  Communications in mathematical physics, 131:347–380.
- Kawamoto, N., Namikawa, Y., Tsuchiya, A., and Yamada, Y. (1988).
  Geometric realization of conformal field theory on riemann surfaces.
  Communications in mathematical physics, 116(2):247–308.
- Knizhnik, V. and Zamolodchikov, A. (1984).
  Current algebra and wess-zumino model in two dimensions.
  Nuclear Physics B, 247(1):83–103.
- Laszlo, Y. (1998).
  Hitchin's and wzw connections are the same.
  Journal of Differential Geometry, 49(3):547–576.
- Laszlo, Y. and Sorger, C. (1997).
  The line bundles on the moduli of parabolic g-bundles over curves and their sections.
  In *Annales scientifiques de l'Ecole normale supérieure*, volume 30, pages 499–525.

References IV

Tsuchiya, A., Ueno, K., and Yamada, Y. (1989).
Conformal field theory on universal family of stable curves with gauge symmetries.
In *Integrable Sys Quantum Field Theory*, pages 459–566. Elsevier.

Ueno, K. (2008).
Conformal field theory with gauge symmetry, volume 24.
American Mathematical Soc.

Wess, J. and Zumino, B. (1974).
Supergauge transformations in four dimensions.
Nuclear Physics B, 70(1):39–50.

Witten, E. (1989).
Quantum field theory and the Jones polynomial.
Communications in Mathematical Physics, 121(3):351–399.